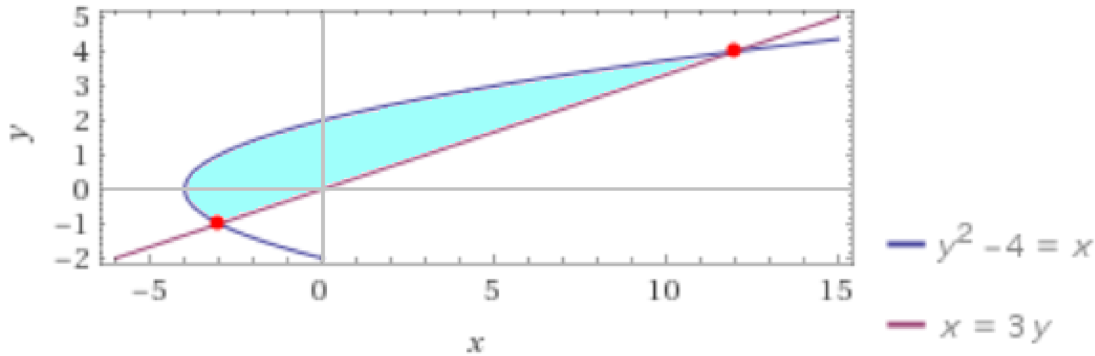
MATH-166
Calculus II

Name: __________KEY__________

1.) Find the area between the curves $y^2 = x + 4$ and $x = 3y$.
Graphing the Function to get a good idea of how it looks we have



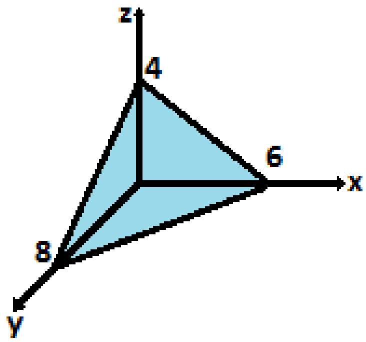
To find the area between the curves we first need to find the points of intersection to get our bounds. Since the x and y values for the equations must agree for the points of intersection. Setting them equal and solving for one of the variables we get

$$\begin{aligned} y^2 - 4 &= 3y \\ y^2 - 3y - 4 &= 0 \\ (y-4)(y+1) &= 0 \end{aligned}$$

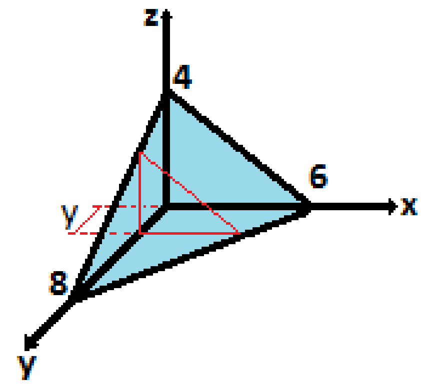
From the above we see our points of intersection are $(12, 4)$ and $(-3, -1)$. Noticing that if we integrate along the x-axis we will have to do two integrals, while on the y-axis we will only have to do one it is easier to do just that. While integrating along the y-axis, we have the positive x-axis is the one to the right. Accordingly, we get the following equation for the area between the curves.

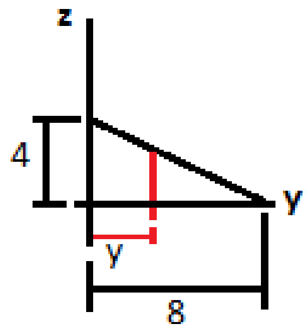
$$\begin{aligned} \text{Area} &= \int_{-1}^{4} (3y) - (y^2 - 4)\ dy \\ &= \int_{-1}^{4} -y^2 + 3y + 4\ dy \\ &= \left[\frac{-y^3}{3} + \frac{3y^2}{2} + 4y \right|_{y=-1}^{4} \\ &= \left(\frac{-4^3}{3} + \frac{3 \cdot 4^2}{2} + 4 \cdot 4 \right) - \left(\frac{-(-1)^3}{3} + \frac{3(-1)^2}{2} + 4(-1) \right) \\ &= \frac{125}{6} = 20.8\overline{3} \end{aligned}$$

2.) Find the volume of the wedge in the figure below using integration.

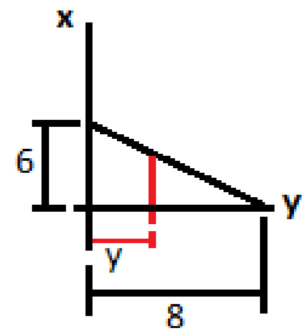


To do this, first pick an axis to integrate along. It's all the same in the end here so I'll just choose y for no particular reason for choosing this axis. This gives us the following pictures





Left side



Bottom

In case it isn't clear from the labels on the graphs, the middle is the triangle on the left side and the right one is the triangle on the bottom of the object. The reason for looking at these sides in particular is that we need the height and width of the red triangular cross section to find the area of the object. To find these values we can exploit the similarity in the triangles on on the left and bottom of the object. So from the side we see the height of the triangle at the red slice is $\frac{4}{8}(8-y)$ and the bottom we see the width is $\frac{6}{8}(8-y)$. If you are unconvinced, try deriving these using properties of triangles.

Then the area, A, of the red triangle is then $A(y) = \frac{1}{2} \cdot \frac{6}{8}(8-y) \cdot \frac{4}{8}(8-y) = \frac{6 \cdot 4}{2 \cdot 8^2}(8-y)^2$. Integrating the area along the y axis gives

$$\begin{aligned}
\text{Volume} &= \int_0^8 A(y)\, dy \\
&= \int_0^8 \frac{6 \cdot 4}{2 \cdot 8^2}(8-y)^2\, dy \\
&= \frac{6 \cdot 4}{2 \cdot 8^2}\left[-\frac{(8-y)^3}{3} \right|_{y=0}^{8} \\
&= \frac{-6 \cdot 4}{3 \cdot 2 \cdot 8^2}(0 - 8^3) \\
&= \frac{8 \cdot 6 \cdot 4}{6} = 32
\end{aligned}$$